Section 20.1: Makespan Minimization

Problem Definition

Schedule: assign each job to one machine.

- each job j has a length $\ell_j > 0$
- load of machine i in a schedule = sum of lengths of jobs assigned to it.
- makespan of schedule = maximum machine load.

want to minimize

Quiz #1

What are the makespans of the following schedules?
(Jobs are labeled with their lengths.)

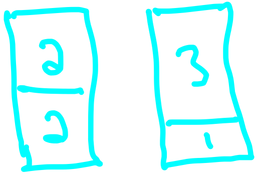
(a) 4 and 3

(b) 4 and 4

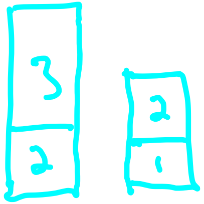
(c) 4 and 5

(d) 8 and 8

Schedule #1



Schedule #2



The Greedy Algorithm Design Paradigm

Input: n jobs with positive lengths $\ell_1, \ell_2, \ldots, \ell_n$ and m machines.

Output: Assignment of jobs to machines that minimizes the makespan.

Greedy algorithm: Construct a solution iteratively, via a sequence of myopic decisions, + hope everything works out in the end.

Graham's Algorithm

Given: m machines, job lengths $\ell_1, \ell_2, \ldots, \ell_n > 0$.

Graham's algorithm:

- do a single pass over jobs (in arbitrary order)
- assign a job j to the machine with the minimum load at the time (given previous assignments of jobs $1, 2, \ldots, j-1$).

Running Time

m = # of machines
n = # of jobs

Straightforward implementation: $O(mn)$

Heap-based implementation: $O(n \log m)$.

Quiz #2

Input = 5 machines, 20 length-1 jobs followed by 1 length-5 job. What are (i) the makespan of the Graham's algorithm schedule; (ii) the min-possible makespan?

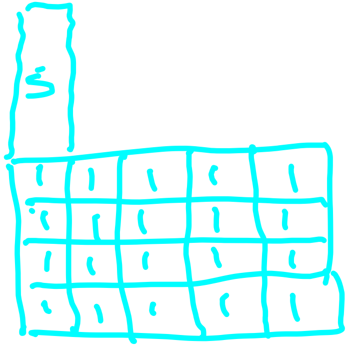
(a) 5 and 4

(b) 6 and 5

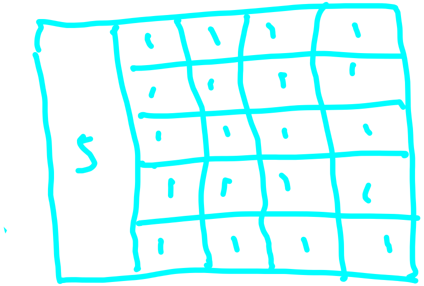
(c) 9 and 5

(d) 10 and 5

Graham



Optimal



Approximate Correctness (Intuition)

Approximate Correctness of Graham's algorithm

m = # of machines

makespan of Graham alg's schedule $\leq \left(2 - \frac{1}{m}\right) \cdot$ minimum-possible makespan.

Intuition:

(1) Smallest machine load $\leq$ min-possible makespan

$\leq$ min-possible makespan

(2) In Graham, (max load − min load) $\leq$ length of a single job

(3) ⟹ max machine load $\leq$ 2 × min-possible makespan

Approximate Correctness (Proof Part 1)

Notation: M^* = min possible makespan
M = makespan of Graham alg's schedule

m = # of machines
$\ell_1, \ell_2, \ldots, \ell_n$ = job lengths

Lowerbound #1: $M^* \geq \ell_j$ for every job j.

Lower bound #2: $M^* \geq \underbrace{\frac{1}{m} \sum_{j=1}^{n} \ell_j}$ ideal load (if perfectly balanced)

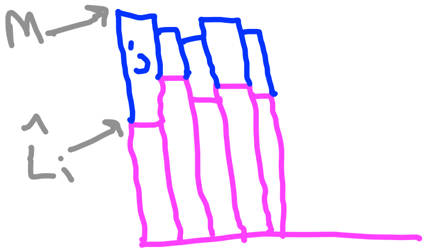
(sum of machine loads = sum of job lengths)

Approximate Correctness (Proof Part 2)

Notation: i = machine with largest load in Graham's schedule (i.e., load $= M$)

j = last job assigned to machine i by Graham's alg.

$\hat{L}_i$ = i's load just before j was assigned



Note: $\hat{L}_i \le \underbrace{\frac{1}{m}\sum_{h=1}^{j-1} \ell_h}_{\text{if perfectly balanced}}$

Thus:

$$M = \hat{L}_i + \ell_j \le \ell_j + \frac{1}{m}\sum_{h=1}^{j-1} \ell_h$$

$$\le \ell_j + \frac{1}{m}\sum_{h \ne j} \ell_h$$

$$= \left(1-\frac{1}{m}\right)\underbrace{\ell_j}_{\le M^* \text{ by LB\#1}} + \underbrace{\frac{1}{m}\sum_{h=1}^{n} \ell_h}_{\le M^* \text{ by LB\#2}}$$

$$\le \left(2-\frac{1}{m}\right)M^*.$$

QED!

M^* = min-possible makespan

M = Graham alg's makespan

Longest Processing Time First (LPT)

LPT Algorithm:

1. sort jobs in nonincreasing order of length
2. run Graham's algorithm

Running time: $O(\underbrace{n\log n}_{\text{using e.g. MergeSort}} + \underbrace{n\log m}_{\text{using heaps}})$

Quiz #3

Input

- 5 machines
- 3 length-5 jobs
- 2 length-6 jobs
- 2 length-7 jobs
- 2 length-8 jobs
- 2 length-9 jobs

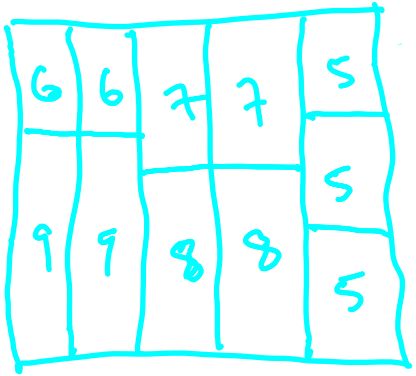
Question: what are (i) makespan of LPT's schedule? (ii) min-possible makespan?

(a) 16 and 15

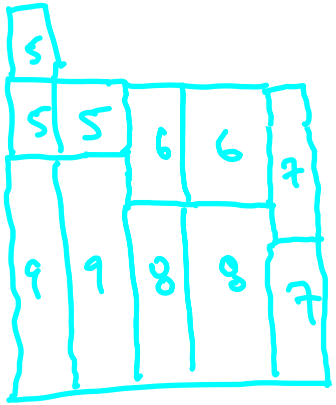
(b) 17 and 15

(c) 18 and 15

(d) 19 and 15



optimal



LPT

Approximate Correctness of LPT

Claim: for LPT, $M \le \left(\frac{3}{2} - \frac{1}{2m}\right) \cdot M^*$.

M = makespan of LPT
M^* = min-possible makespan

can be improved to $\left(\frac{4}{3} - \frac{1}{3m}\right)$ (exercise)

LB#1 (new version): If j not among the m longest jobs, $M^* \ge 2\ell_j$.

[reason: every schedule puts 2 of the longest $m+1$ jobs on a common machine (Pigeonhole Principle)]

Pf of Claim: can assume ≥ 2 jobs assigned to i.

i = most loaded machine in LPT's schedule
j = last job assigned to i

$$M \le \left(1 - \frac{1}{m}\right) \ell_j + \frac{1}{m} \sum_{h=1} \ell_j \le \left(\frac{3}{2} - \frac{1}{2m}\right) M^*.$$

$\le M^*/2$ (LB#1) $\le M^*$ (LB#2)

QED!