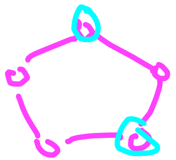
Section 22.4: Independent Set Is NP-Hard

The Independent Set Problem



Input: undirected graph $G=(V,E)$.

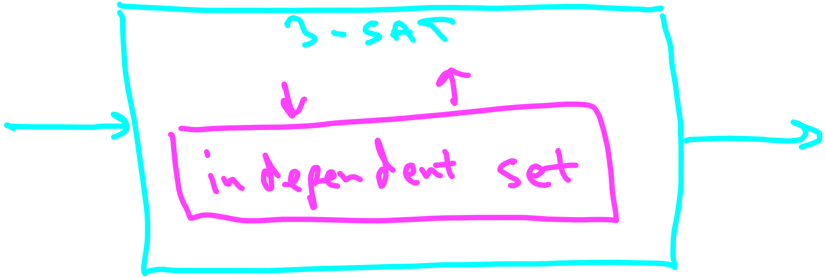
mutually non-adjacent vertices

Output: independent set $S \subseteq V$ of max-possible size.

Theorem: 3-SAT reduces to the independent set problem.

Corollary: independent set is an NP-hard problem. [assuming Cook-Levin theorem]

The Plan



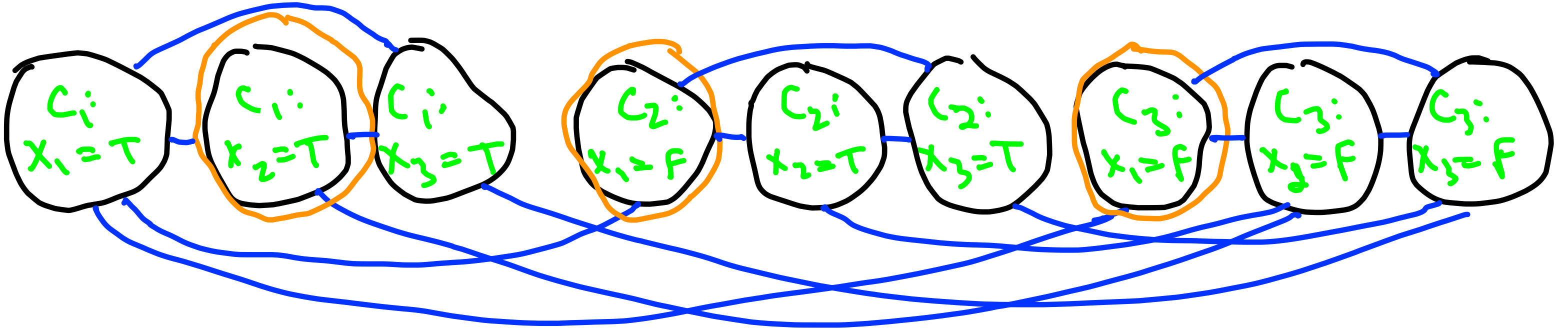
Note: need to translate given 3-SAT instance into an undirected graph. (?)

(should be able to extract solution to 3-SAT instance from max-size independent set)

Constraints as Assignment Requests

Key idea: introduce one variable per assignment request. $\neg x_1 \vee x_2 \vee x_3$

Example: $\underbrace{x_1 \vee x_2 \vee x_3}_{c_1}$ $\underbrace{\neg x_1 \vee x_2 \vee x_3}_{c_2}$ $\underbrace{\neg x_1 \vee \neg x_2 \vee \neg x_3}_{c_3}$



$\Rightarrow$ satisfying assignments F/T/F & F/T/T.

The Reduction

Given: instance of 3-SAT (n variables, m constraints).

Construct: graph $G=(V,E)$ where

- for $i=1,2,\ldots,m$, V_i = set of k_i vertices [k_i = # of literals in its constraint]
- $V = V_1 \cup V_2 \cup \cdots \cup V_m$
- E_1 = edge for each vertex pair in same group V_i
- E_2 = edge for each pair of conflicting vertices

$E = E_1 \cup E_2$

Compute: max-size independent set S of G. (using assumed subroutine)

Return: (i) if $|S| = m$, return truth assignment consistent with S (ii) if not, return "unsatisfiable"

The Reduction: Correctness Proof

① edges of E_1 ⇒ an independent set has ≤ 1 vertex per group. (⇒ max size $\leq m$)

② edges of E_2 ⇒ exists a truth assignment consistent with each independent set.

③ S has size m ⇒ one vertex per group
⇒ any consistent truth assignment satisfies every constraint

Case 1: [3-SAT instance is satisfiable] pick one satisfied assignment request per constraint ⇒ indep set of size m.

Case 2: [3-SAT instance is unsatisfiable] can't be any independent set of size m [if there were, could extract satisfying assignment]

QED

Quiz

Where does the proof break down if the intragroup edges E_1 are omitted from the graph G?

(a) An independent set of G no longer translates to a well-defined truth assignment.

(b) A satisfiable 3-SAT instance need not translate to a graph with max-size independent set $\geq m$.

(c) An unsatisfiable 3-SAT instance need not translate to a graph with max-size independent set $< m$.

(d) Actually, the proof still works!