Section 22.5: Directed Hamiltonian Path Is NP-Hard

The Directed Hamiltonian Path Problem

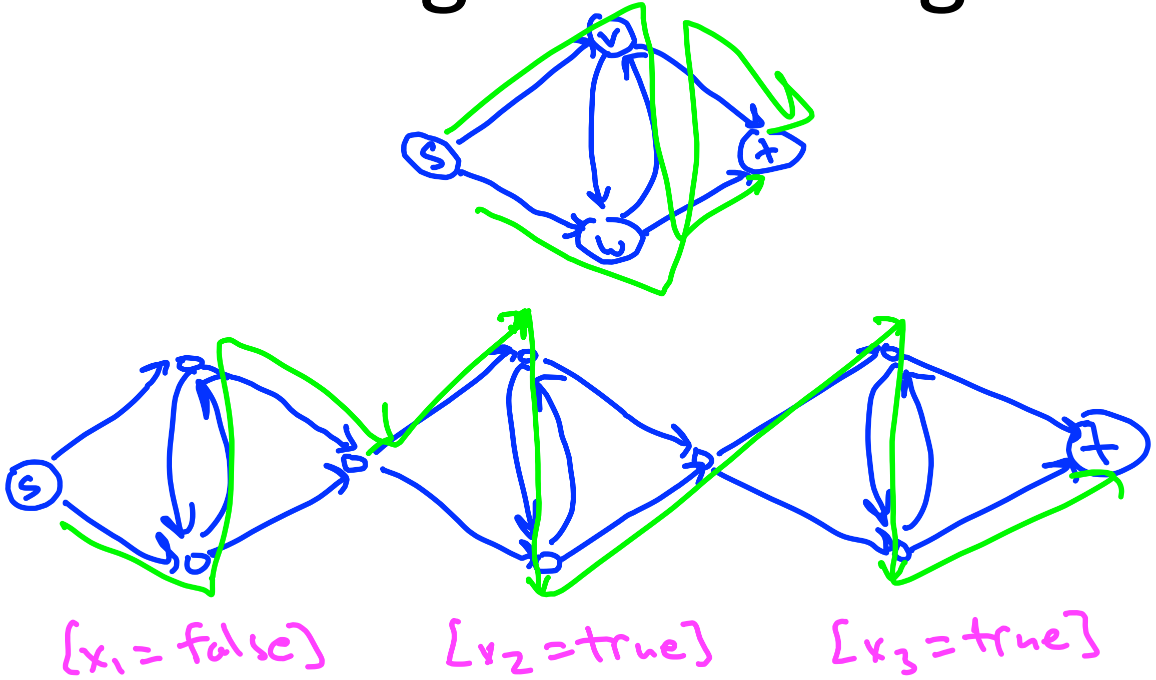
Input: directed graph $G=(V,E)$, starting vertex $s \in V$, destination $t \in V$.

Output: an s-t Hamiltonian path (visits every vertex exactly once).
[or, if no such paths, correctly report this fact]

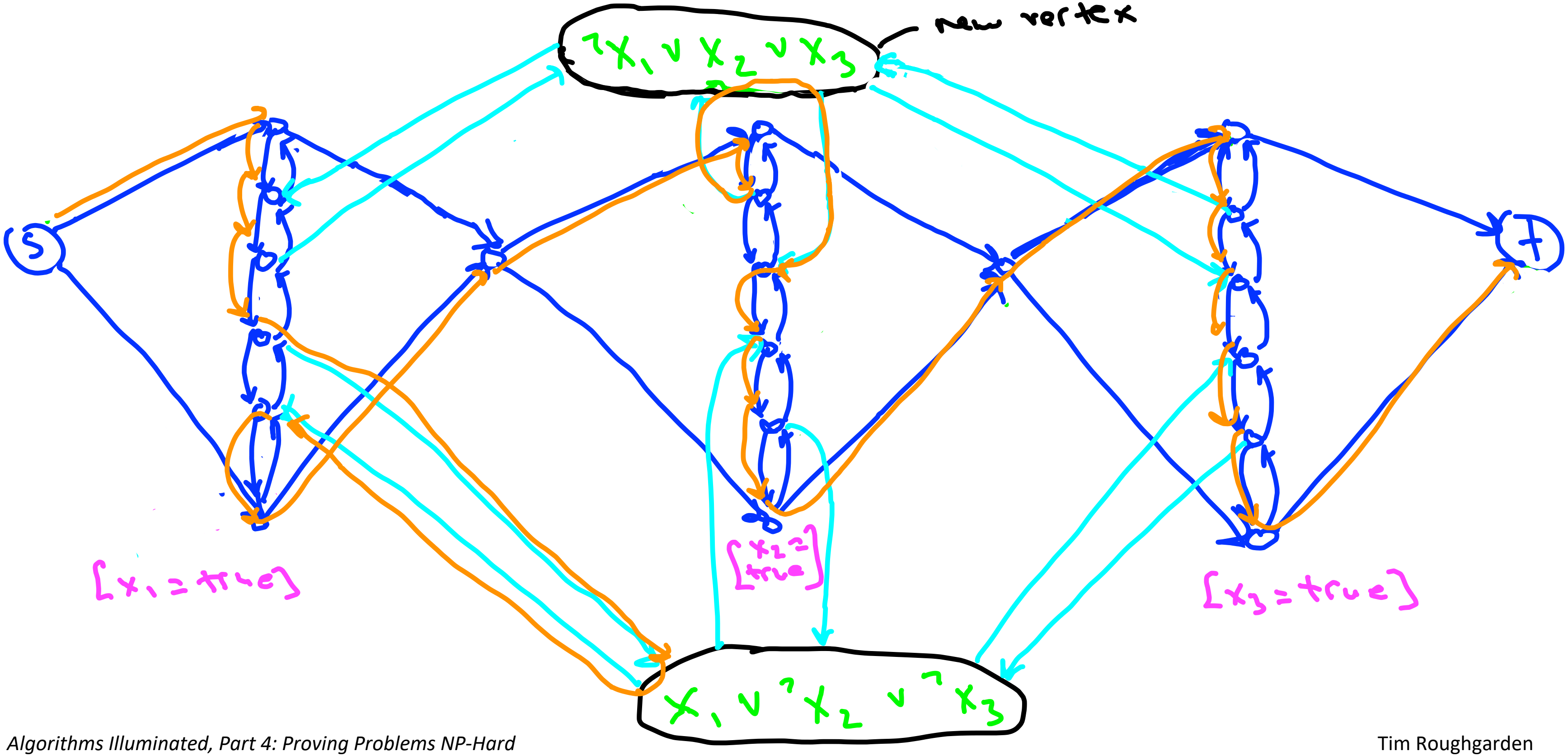
Theorem: 3-SAT reduces to directed Hamiltonian path.

Corollary: directed Hamiltonian path is an NP-hard problem.
[assuming the Cook-Levin theorem]

Encoding Truth Assignments



Encoding Constraints

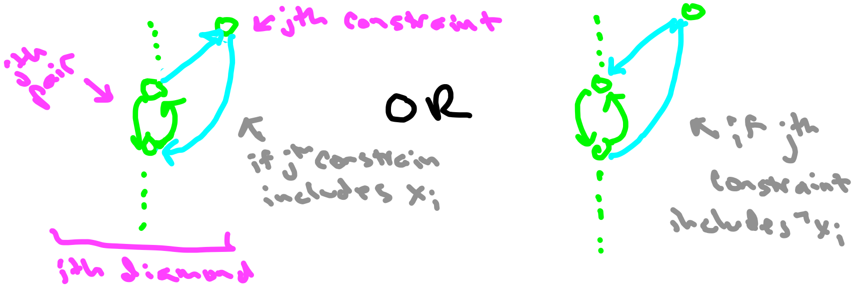


The Reduction

Given: instance of 3-SAT. [n variables, m constraints]

Construct: a directed graph G with:

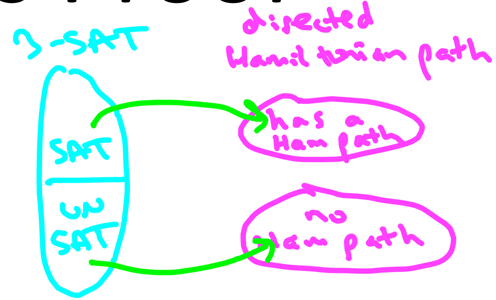
- n diamonds, upward/downward internal paths of length 2m+1
- one vertex per constraint [2nm + 3n + m + 1 vertices in all]
- edges inside necklace
- wire necklace to constraint vertices



Compute: Hamiltonian path of G (if one exists).

Return: (i) given Hamiltonian path of G, return corresponding truth assignment (ii) if G has no Hamiltonian path, return "unsatisfiable"

The Reduction: Correctness Proof



Observation: every Hamiltonian path of G traverses diamonds in sequence up or down, interrupted only by back+forth trips to constraint vertices.

Also: can visit a constraint vertex from diamond ⟺ satisfying one of its assignment requests.

Therefore: can extract satisfying assignment from a Hamiltonian path.

Case 1: [3-SAT instance satisfiable] ⟹ G has a Hamiltonian path, subroutine will return one.

Case 2: [unsatisfiable] G can't have any Hamiltonian path, subroutine will report this.

QED!