Section 22.6: TSP Is NP-Hard

The Undirected Hamiltonian Path Problem

Input: undirected graph $G=(V,E)$, starting vertex s, destination t.

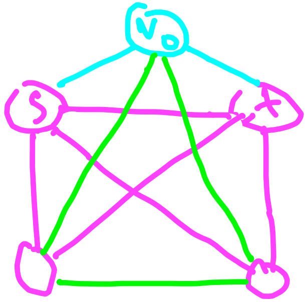
Output: an s-t Hamiltonian path of G. [visits every vertex exactly once]
(or correctly report that none exist.)

Exercise: undirected/directed versions reduce to each other.
⇒ undirected Hamiltonian path also NP-hard

Theorem: undirected Hamiltonian path reduces to the TSP.

Corollary: The TSP is NP-hard.

The Reduction



Given: instance ($G=(V,E)$) of undirected Hamiltonian path.

Construct: graph G', edge costs c.

- add vertex v_0, edges (v_0,s), (v_0,t)
- give all edges cost $c_e=0$
- fill in missing edges, give them cost $c_e=1$.

Compute: min-cost tour of G'. [using assumed subroutine]

Return: (i) if given tour T with cost 0, return s-t path $T-\{(v_0,s),(v_0,t)\}$.

(ii) if cost is >0, return "no Hamiltonian path in G"

The Reduction: Correctness Proof

(1) zero-cost tour $T \Rightarrow$ must use (v_0, s) & (v_0, t).

(2) zero-cost tour $T \Rightarrow T - \{(v_0,s), (v_0,t)\}$ a Hamiltonian path of G.

Case 1: [G has a Hamiltonian path]

$\Rightarrow$ G' has a zero-cost tour, subroutine will return a zero-cost tour.

Case 2: [G has no Hamiltonian path]

$\Rightarrow$ G' has no zero-cost tours, subroutine will correctly detect this

QED!